

Time-delayed feedback control of a flashing ratchet

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Closed-loop or feedback control ratchets use information about the state of the system to operate with the aim of maximizing the performance of the system. In this paper we investigate the effects of a *time delay* in the feedback for a protocol that performs an instantaneous maximization of the center-of-mass velocity. For the *one* and the *few particle* cases the flux decreases with increasing delay, as an effect of the decorrelation of the present state of the system with the information that the controller uses, but the delayed closed-loop protocol succeeds to perform better than its open-loop counterpart provided the delays are smaller than the characteristic times of the Brownian ratchet. For the *many particle* case, we also show that for small delays the center-of-mass velocity decreases for increasing delays. However, for large delays we find the surprising result that the presence of the delay can improve the performance of the nondelayed feedback ratchet and the flux can attain the maximum value obtained with the optimal periodic protocol. This phenomenon is the result of the emergence of a dynamical regime where the presence of the delayed feedback stabilizes one quasiperiodic solution or several (multistability), which resemble the solutions obtained in the so-called threshold protocol. Our analytical and numerical results point towards the feasibility of an *experimental implementation* of a feedback controlled ratchet that performs equal or better than its optimal open-loop version.

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I. INTRODUCTION

The ratchet effect consists of the emergence of a directed transport in a spatially periodic system out of equilibrium through the introduction of an external perturbation. The celebrated ideas of rectifying thermal noise, originally introduced by Smoluchowski [1] and later resumed by Feynman [2], were explicitly used in the context of directed transport in the 1990s [3, 4, 5]. Since then, these systems have been studied due to its importance from a theoretical point of view in nonequilibrium physics [6] and its applications to many other fields such as condensed matter or biology [6, 7].

One of the main ratchet types are the flashing ratchets that operate switching on and off a spatially periodic asymmetric potential. A simple periodic or random switching is able to achieve a rectification of thermal fluctuations and produce a net current of particles. Recently, a new class of control protocols that use instant information about the state of the system to take the decision of switching on or off have been introduced [8]. These so-called closed-loop or feedback control protocols have been proven to be an effective way to increase the net current in collective Brownian ratchets [8, 9, 10]. Feedback

control can be implemented in systems where particles are monitored [11, 12]. This monitoring gives information about the position of the particles that can be used to switch on or off the potential in real time according to a given protocol. For instance, in Ref. [11] the motion of colloidal particles induced by a sawtooth dielectric potential, which is turned on and off periodically, is experimentally studied monitoring the particles. This suggests that a feedback controlled version of the ratchet in [11] can be constructed gathering information about the state of the system with a charge coupled device (CCD) camera and using this information to decide whether to turn on and off the potential in real time. In addition, feedback ratchets have been recently suggested as a mechanism to explain the stepping motion of the two-headed kinesin [13].

All Brownian feedback ratchets considered until now use instant information to operate, that is, they all measure the state of the system and act *instantaneously* according to that measurement. However, in realistic devices there is always a time delay between the input measurements and the output control action due to physical limitations to the velocity of transmission and processing of the information [14, 15]. For example, in the construction of the feedback controlled version of the ratchet in [11] time delays in the feedback will be present due to the finite time needed to take a picture with a CCD camera, transmit it, process it, and implement the resulting decision of switching on or off the potential. Therefore it

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is important to compute the effects of time delays in the feedback, because it clarifies in which real ratchet systems it is experimentally feasible to obtain the increase of velocity predicted in [8]. The study of time-delayed feedback is also relevant because it appears naturally in many stochastic processes, such as complex systems with self-regulating mechanisms [16, 17]. For another type of ratchets, deterministic feedback ratchets, some of the effects of time delay in the feedback have been studied [18, 19].

In the current paper we investigate how a time delay in the control of a feedback flashing ratchet affects the net flux. In the next section we describe the ratchet model with the time-delayed feedback control policy. In Sec. III we study in detail the case of one particle, getting an effective potential description for the flux in the relevant case of small time delays. We also present an alternative approach to understanding the dependence of the flux with time delay in terms of the covariance, and we describe the behavior for large delays. In Sec. IV we treat the collective ratchet with few particles and relate its center-of-mass velocity with the one particle flux previously studied. In Sec. V we study the many particle ratchet, which exhibits a somehow counterintuitive behavior; first we briefly review the results for zero delays that will be useful, and thereafter we expose the results in the two dynamical regimes of small delays and large delays. Finally, all the results are summarized and discussed in Sec. VI.

II. MODEL

We consider N overdamped Brownian particles at temperature T in a ratchet potential $V(x)$. The force acting on the i th particle at position $x_i(t)$ is $\alpha(t)F(x_i(t))$, where $F(x) = -V'(x)$ and $\alpha(t)$ implements the action of the controller. Therefore the system dynamics is defined by the Langevin equations

$$\gamma \dot{x}_i(t) = \alpha(t)F(x_i(t)) + \xi_i(t); \quad i = 1, \dots, N, \quad (1)$$

where γ is the friction coefficient (related to the diffusion coefficient D through Einstein's relation $D = k_B T / \gamma$) and $\xi_i(t)$ are Gaussian white noises of zero mean and variance $\langle \xi_i(t) \xi_j(t') \rangle = 2\gamma k_B T \delta_{ij} \delta(t - t')$.

In order to study the effects of time-delayed feedback controls let us include a time delay of value τ in the control of the paradigmatic *maximization of the center-of-mass instant velocity* protocol [8]. The controller measures the sign of the net force per particle

$$f(t) = \frac{1}{N} \sum_{i=1}^N F(x_i(t)), \quad (2)$$

and, after a time τ , it switches on the potential ($\alpha = 1$) if the net force was positive or it switches off ($\alpha = 0$) if

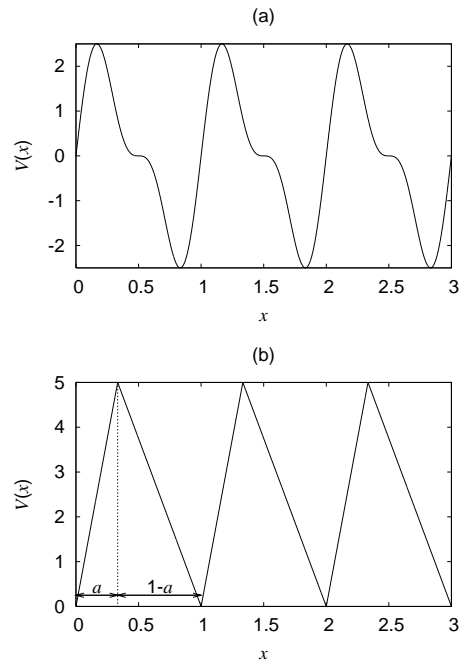


FIG. 1: Panel (a): ‘Smooth’ potential (5) for $V_0 = 5k_B T$. Panel (b): ‘Sawtooth’ potential (4) for $V_0 = 5k_B T$ and $a = 1/3$. Units: $L = 1$, $k_B T = 1$.

it was negative. Thus the control protocol reads

$$\alpha(t) = \begin{cases} \Theta(f(t - \tau)) & \text{if } t \geq \tau, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

with Θ the Heaviside function [$\Theta(x) = 1$ if $x > 0$, else $\Theta(x) = 0$].

Finally, to completely fix the model we choose a piecewise linear sawtooth potential $V(x) = V(x + L)$ of height V_0 and asymmetry parameter $a < 1/2$,

$$V(x) = \begin{cases} \frac{V_0}{a} \frac{x}{L} & \text{if } 0 \leq \frac{x}{L} \leq a, \\ V_0 - \frac{V_0}{1-a} \left(\frac{x}{L} - a \right) & \text{if } a < \frac{x}{L} \leq 1. \end{cases} \quad (4)$$

We have verified that the results found in this paper are valid for other potentials provided they have the same height of the potential V_0 and the same asymmetry parameter a , with V_0 defined as the difference between the maximum and the minimum values of the potential and aL as the distance between the minimum and the maximum positions. For this verification we have considered the ‘smooth’ potential

$$V(x) = \frac{2V_0}{3\sqrt{3}} \left[\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{2} \sin\left(\frac{4\pi x}{L}\right) \right], \quad (5)$$

which has potential height V_0 , period L , and asymmetry $a = 1/3$. See Fig. 1.

In the study of these feedback ratchets it proves to be useful to distinguish three cases: one particle, few particles, and many particles. This classification is based

on the results of the zero delay studies [8, 9, 10], which revealed different characteristics and analytical approximations for each case. The many particle case is formed by those feedback collective ratchets that for zero delay have net force fluctuations smaller than the maximum absolute value of the net force; see Refs. [8, 9, 10].

Throughout the rest of this paper, we will use units where $L = 1$, $k_B T = 1$, and $D = 1$.

III. ONE PARTICLE

In this section we discuss the simpler case of a ratchet consisting of one particle, so that the position $x(t)$ is governed by Eq. (1) with $N = 1$ and $\alpha(t)$ given by Eq. (3), which is a nonlinear stochastic delay differential equation. In general, there is no analytical treatment for these time-delayed stochastic equations. Here, we shall write the corresponding delay Fokker-Planck equation [20], and use a perturbative technique [17, 21] to obtain an effective potential description for small delays that leads to approximate analytical expressions for the flux. Finally, in this section we shall get insight in the regime of large delays by studying the covariance of the sign of the net force.

The force that the particle feels with the inclusion of the time delay τ in the control [Eq. (3)] depends both on the actual position $x := x(t)$ and on the delayed position $x_\tau := x(t - \tau)$. This force $F_\tau(x, x_\tau)$ is periodic in both arguments, $F_\tau(x, x_\tau) = F_\tau(x + 1, x_\tau) = F_\tau(x, x_\tau + 1)$, and reads

$$F_\tau(x, x_\tau) = \begin{cases} 0 & \text{if } 0 \leq x_\tau \leq a, \\ \frac{-V_0}{a} & \text{if } a < x_\tau \leq 1 \text{ and } 0 \leq x \leq a, \\ \frac{V_0}{1-a} & \text{if } a < x_\tau \leq 1 \text{ and } a < x \leq 1. \end{cases} \quad (6)$$

In particular, $F_\tau(x, x) =: F_0(x)$ corresponds to the effective force of the instant maximization control protocol

without delay [8], i.e.,

$$F_0(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a, \\ \frac{V_0}{1-a} & \text{if } a < x \leq 1. \end{cases} \quad (7)$$

In terms of the force (6), the evolution of the position of the particle obeys the stochastic delay differential equation

$$\dot{x}(t) = F_\tau(x(t), x(t - \tau)) + \xi(t). \quad (8)$$

The probability density $\rho(x, t)$ of this stochastic process satisfies a delay Fokker-Planck equation [17, 20, 21, 22], which involves the two-point probability density as follows:

$$\frac{\partial}{\partial x} \rho(x, t) = -\frac{\partial}{\partial x} \int F_\tau(x, x_\tau) \rho(x, t; x_\tau, t - \tau) dx_\tau + \frac{\partial^2}{\partial x^2} \rho(x, t). \quad (9)$$

For small delays, this equation can be treated perturbatively; then, following Refs. [17, 21], the explicit effective force for small delays can be achieved by computing

$$F_{\text{eff}}(x) = \int F_\tau(x, x_\tau) P(x_\tau, t + \tau | x, t) dx_\tau, \quad (10)$$

where the short time propagator $P(x, t + \tau | x, t)$ (see §4.4.1 in Ref. [23]) is

$$P(x_\tau, t + \tau | x, t) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{[x_\tau - x - F_0(x)\tau]^2}{2\tau}\right). \quad (11)$$

Due to the Gaussian form of the propagator in this small delay approximation, we can neglect the long tails of the Gaussian propagator and restrict the integration in Eq. (10) to the intervals $(a - 1, 1)$ and $(0, 1 + a)$ for $0 \leq x \leq a$ and $a < x \leq 1$, respectively. We get

$$F_{\text{eff}}(x) = F_{\text{eff}}(x + 1) = \begin{cases} -\frac{V_0}{2a} \left[\text{erfc}\left(\frac{x}{\sqrt{2\tau}}\right) + \text{erfc}\left(\frac{a-x}{\sqrt{2\tau}}\right) \right] & \text{if } 0 \leq x \leq a, \\ \frac{V_0}{2(1-a)} \left[2 - \text{erfc}\left(\frac{1-x-\frac{V_0\tau}{1-a}}{\sqrt{2\tau}}\right) - \text{erfc}\left(\frac{x-a+\frac{V_0\tau}{1-a}}{\sqrt{2\tau}}\right) \right] & \text{if } a < x \leq 1, \end{cases} \quad (12)$$

where $\text{erfc}(x)$ is the complementary error function. On the other hand, the value of the effective force can be computed numerically by splitting in bins the position of the particle and evaluating the probability of being in those bins. For small delays, Eq. (12) gives a good estimation as shown in Fig. 2.

The main effect of the inclusion of a small delay in the control is a slant of the effective force near the points of discontinuity. This effect lies on the idea that the closer

the particle is to the discontinuities, the more probable is that the controller makes a mistake. For instance, when the particle is to the left of $x = a$ and close to it, there are two possibilities: (i) if the retarded position was to the left too then the controller sets the potential off, and (ii) if the retarded position was to the right then the controller sets the potential on and the particle feels a negative force $-V_0/a$. Therefore in the points to the left of $x = a$ and close to it the force takes an effective value

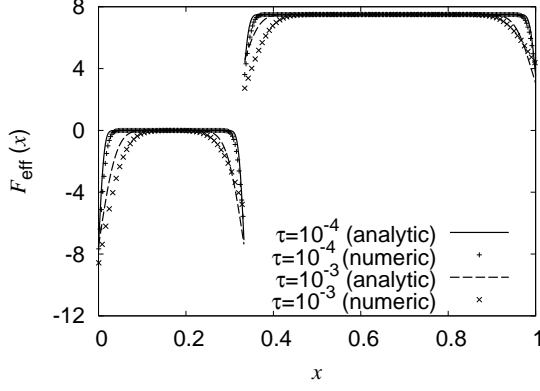


FIG. 2: Effective force for small delays for potential height $V_0 = 5k_B T$ and asymmetry $a = 1/3$ in the one particle case [Eq. (12)]. Units: $L = 1$, $D = 1$, $k_B T = 1$.

between 0 and $-V_0/a$, resulting in a negative effective

$$V_{\text{eff}}(x) = \begin{cases} \frac{V_0 \sqrt{2\tau}}{2a} \left[\text{ierfc}\left(\frac{a}{\sqrt{2\tau}}\right) + \text{ierfc}\left(\frac{x}{\sqrt{2\tau}}\right) - \text{ierfc}\left(\frac{a-x}{\sqrt{2\tau}}\right) - \frac{1}{\sqrt{\pi}} \right] & \text{if } 0 \leq x \leq a, \\ \frac{V_0 \sqrt{2\tau}}{2(1-a)} \left[\frac{2(a-x)}{\sqrt{2\tau}} + \text{ierfc}\left(\frac{1-x-\frac{V_0\tau}{1-a}}{\sqrt{2\tau}}\right) - \text{ierfc}\left(\frac{1-a-\frac{V_0\tau}{1-a}}{\sqrt{2\tau}}\right) + \text{ierfc}\left(\frac{\frac{V_0\tau}{1-a}}{\sqrt{2\tau}}\right) \right. \\ \left. - \text{ierfc}\left(\frac{x-a+\frac{V_0\tau}{1-a}}{\sqrt{2\tau}}\right) + \frac{2(1-a)}{a} \text{ierfc}\left(\frac{a}{\sqrt{2\tau}}\right) - \frac{2(1-a)}{a\sqrt{\pi}} \right] & \text{if } a < x \leq 1, \end{cases} \quad (15)$$

in the interval $[0, 1]$, and outside $V_{\text{eff}}(x) = V_{\text{eff}}(y) + (x - y)V_{\text{eff}}(1)$, with $y \equiv x \bmod 1$, $y \in [0, 1]$. The function ierfc is the first iterated integral of the complementary error function [24],

$$\text{ierfc}(x) = \int_x^\infty \text{erfc}(s) ds = -x \text{erfc}(x) + \frac{e^{-x^2}}{\sqrt{\pi}}. \quad (16)$$

This effective potential is depicted in Fig. 3, where we see that an increase of the delay implies a decrease of the average tilt of the potential. Eventually, the stationary flux is calculated inserting Eq. (15) in Eq. (14). The resulting approximate expression gives good results for very small delays and a good estimate of the decrease rate for small delays. See Fig. 4. [This can be understood noting that although for some positions the corrections to the effective force are appreciable already for quite small delays (see Fig. 2) this only happens in small space intervals and therefore the results for the flux are better than expected.] The approximate analytical expression obtained gives the average velocity in terms of the main magnitudes of the system, namely, the height of the potential V_0 , its asymmetry a , and the time delay in the feedback τ . We have checked that this result is in good

force.

In this effective description the position of the particle evolves with a Langevin equation $\dot{x}(t) = F_{\text{eff}}(x) + \xi(t)$, with the associated (nondelayed) effective Fokker-Planck equation

$$\frac{\partial}{\partial t} \rho(x, t) = -\frac{\partial}{\partial x} [\rho(x, t) F_{\text{eff}}(x)] + \frac{\partial^2}{\partial x^2} \rho(x, t), \quad (13)$$

with periodic boundary conditions. The average velocity is obtained computing the expectation value of the velocity in the stationary distribution of the effective Fokker-Planck equation [6]:

$$\langle \dot{x} \rangle = \frac{1 - e^{V_{\text{eff}}(1) - V_{\text{eff}}(0)}}{\int_0^1 dx \int_x^{x+1} dy e^{V_{\text{eff}}(y) - V_{\text{eff}}(x)}}, \quad (14)$$

where $V_{\text{eff}}(x) = -\int_0^x F_{\text{eff}}(s) ds$. Integrating $F_{\text{eff}}(x)$ we get the expression of the approximate effective potential for small delays $V_{\text{eff}}(x)$,

agreement also for other potentials.

Another approach can be taken to understand the observed decrease in the flux for increasing delay. The instant maximization protocol does not use detailed information about the position of the particles, it simply deals with the sign of the net force, namely, $\text{sgn}f$, [with $\text{sgn}(x) = 1$ for $x > 0$, $\text{sgn}(x) = 0$ for $x = 0$, and $\text{sgn}(x) = -1$ for $x < 0$]. The flux performance of the protocol would be optimal if it would have received the present sign of the net force, $\text{sgn}f(t)$, but it does receive its value a time τ earlier, $\text{sgn}f(t - \tau)$. This earlier value contains information about the present value because both values are correlated, as can be shown computing the covariance

$$\begin{aligned} \tilde{C}(\tau) &:= \langle [\text{sgn}f(t) - \mu][\text{sgn}f(t - \tau) - \mu] \rangle \\ &= C(\tau) - \mu^2, \end{aligned} \quad (17)$$

where

$$\begin{aligned} C(\tau) &:= \langle \text{sgn}f(t) \text{sgn}f(t - \tau) \rangle, \\ \mu &:= \langle \text{sgn}f(t) \rangle = \langle \text{sgn}f(t - \tau) \rangle. \end{aligned} \quad (18)$$

The decrease of the function $\tilde{C}(\tau)$ for increasing τ (Fig. 5)

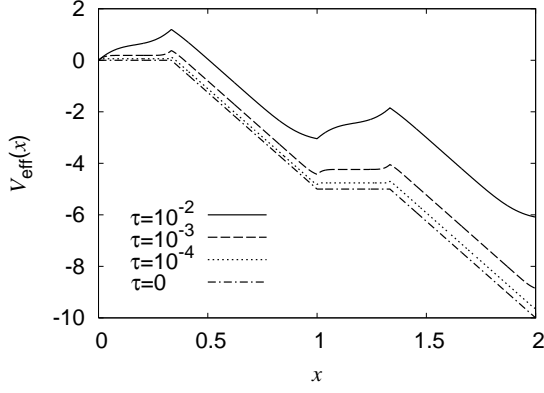


FIG. 3: Effective potential [Eq. (15)] for potential height $V_0 = 5k_B T$ and asymmetry $a = 1/3$ for time delays $\tau = 0, 10^{-4}, 10^{-3}$, and 10^{-2} . Units: $L = 1$, $D = 1$, $k_B T = 1$.

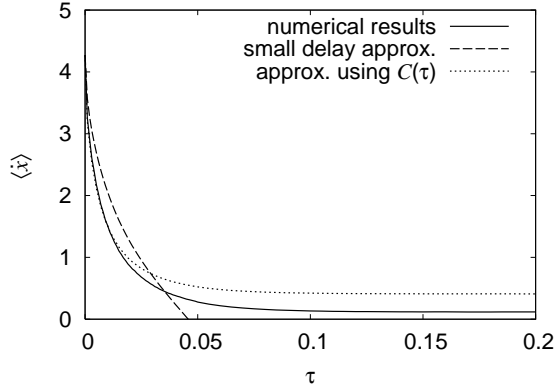


FIG. 4: One particle flux vs delay. The numerical result (solid line) is compared with the estimations using the effective potential (15) (dashed line) and using $C(\tau)$ (dotted line). $V_0 = 5k_B T$ and $a = 1/3$. Units: $L = 1$, $D = 1$, $k_B T = 1$.

explains the decrease of the center-of-mass velocity as a consequence of the loss of information about the present sign of the net force. In addition, we can obtain an estimation of the flux decrease with the following heuristic argument. Let us calculate the covariance using

$$\begin{aligned} C &= P_{++} + P_{--} - P_{+-} - P_{-+}, \\ \mu &= P_{++} - P_{--} + P_{+-} - P_{-+}, \end{aligned} \quad (19)$$

where P_{ij} is the joint probability of having a positive ($i = +$) or negative ($i = -$) net force at time t and a positive ($j = +$) or negative ($j = -$) net force at time $t - \tau$. These probabilities can be computed if we assume that the system performance can be explained with the simplified description that $\text{sgn}f(t - \tau)$ is different from $\text{sgn}f(t)$ with probability p . This description allows us to use the results found in [25] for the instantaneous

maximization protocol with a controller receiving $\text{sgn}f(t)$ through a noisy channel with noise level p . In fact, notice that the plot of the effective potential (Fig. 3) resembles the form of the effective potential found in Ref. [25] for the noisy channel. This elementary description gives the values $P_{-+} = bp$, $P_{--} = b(1 - p)$, $P_{+-} = (1 - b)p$ and $P_{++} = (1 - b)(1 - p)$ for the joint probabilities, with b the probability of $\text{sgn}f(t)$ being negative. Thus Eqs. (19) can be rewritten in terms of the probability of error $p = P_{+-} + P_{-+}$ and the probability $b = P_{-+} + P_{--}$ as

$$\begin{aligned} C &\approx 1 - 2p, \\ \mu &\approx 1 - 2b. \end{aligned} \quad (20)$$

In [25] it is shown that for small potential heights (small V_0) $b \approx a$ and

$$\langle \dot{x} \rangle \approx V_0(1 - 2p). \quad (21)$$

Therefore this simplified description suggests

$$\langle \dot{x} \rangle \sim V_0 C. \quad (22)$$

For larger potential heights, a better estimation is obtained evaluating the general expression $\langle \dot{x} \rangle(p)$ of Ref. [25] at $p(\tau) = [1 - C(\tau)]/2$. This estimation is plotted in Fig. 4, where it is compared with numerical results and the analytical small delay approximation [Eqs. (14) and (15)].

The average velocity of the particle for large delays is not zero, but reaches a constant value independent of the delay (see Fig. 4). We have seen that the function $C(\tau)$ also tends to a constant nonzero value in the same characteristic time that the velocity does, in qualitative agreement with the estimation described after Eq. (22), although this estimation does not give the correct value of the flux. Therefore this estimation gives good quantitative results for small delays and the qualitative behavior for large delays. The large τ behavior observed for the flux implies an effective force independent of the time delay τ for large enough values of τ , as we show in Fig. 6. The average over x of the numerical large τ effective force is positive, in agreement with the positive net flux obtained. For example, for asymmetry parameter $a = 1/3$ and potential heights $V_0 = 1, 5$, and 10 , the net flux is $\langle \dot{x} \rangle_{\tau \rightarrow \infty} \approx 0.01, 0.12$, and 0.18 , respectively. The convergence to this constant value can be understood realizing that the covariance \tilde{C} becomes negligible for large delays, i.e., the fluctuations of $\text{sgn}f$ around its mean value at t and at $t - \tau$ are independent. This indicates that the system dynamics is effectively the same as that for an open-loop control protocol, as the correlation between the switches and the state of the system are negligible.

Comparing the results for the delayed instant maximization protocol with the optimal periodic open-loop protocol, we see that the former performs better than the latter even for nonzero delay, provided the delay is

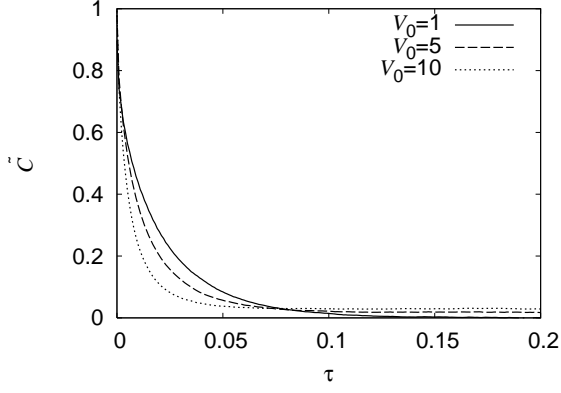


FIG. 5: Covariance \tilde{C} [Eq. (17)] as a function of the time delay for potential heights $V_0 = k_B T$, $5k_B T$, and $10k_B T$. Asymmetry parameter $a = 1/3$. Units: $L = 1$, $D = 1$, $k_B T = 1$.

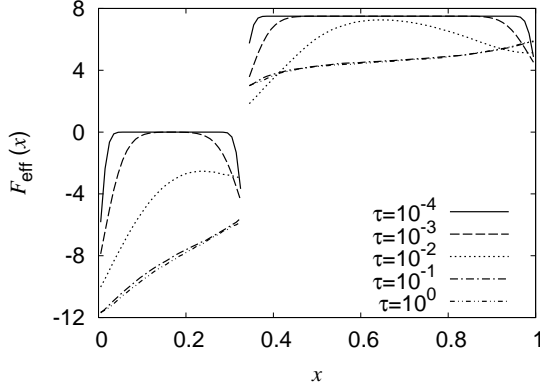


FIG. 6: Effective force (from numerical simulations) ($N = 1$) for several time delays. $V_0 = 5k_B T$ and $a = 1/3$. Units: $L = 1$, $D = 1$, $k_B T = 1$.

smaller than the characteristic times of the dynamics of the Brownian ratchet. Therefore the instant maximization protocol gives a larger flux than the optimal open-loop control protocol for time delays τ such that $\tau \ll \mathcal{T}_{\text{on}}$, $\tau \ll \mathcal{T}_{\text{off}}$, where $\mathcal{T}_{\text{on}} \sim (1-a)^2/V_0$ and $\mathcal{T}_{\text{off}} \sim a^2/2$ are the on-potential and off-potential times in the optimal periodic protocol [5]. See, for example, Fig. 4 and compare with $\langle \dot{x} \rangle_{\text{open}} \approx 0.3$ that is the value for the optimal periodic protocol for $V_0 = 5$ and $a = 1/3$, which has $\mathcal{T}_{\text{on}} \approx 0.06$ and $\mathcal{T}_{\text{off}} \approx 0.05$.

IV. FEW PARTICLES

In this section we deal with a collective ratchet compounded of a few particles. We will show that the center-of-mass velocity for the few particle case can be related with the velocity obtained for the one particle ratchet

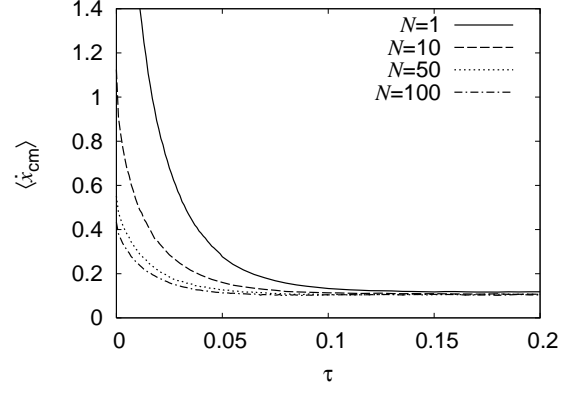


FIG. 7: Center-of-mass velocity as a function of the time delay for number of particles $N = 1, 10, 50$, and 100 . Parameters of the potential: $V_0 = 5k_B T$ and $a = 1/3$. Units: $L = 1$, $D = 1$, $k_B T = 1$.

studied in the section before.

As in the one particle case, the effect of the inclusion of a delay is a decrease in the covariance and in the center-of-mass velocity (Fig. 7). Therefore we can also interpret the decrease in the center-of-mass velocity as a consequence of the loss of information of the present sign of the net force, and then assume that the system performance can be explained with the simplified description that $\text{sgn}f(t - \tau)$ is different from $\text{sgn}f(t)$ with probability p . This simplified description leads for small potential heights [25, 26] to

$$\begin{aligned} C_N(\tau) &\sim 1 - 2p_N, \\ \mu_N &\sim 1 - 2b_N, \\ \langle \dot{x}_{\text{cm}} \rangle_N &\approx \frac{V_0(1 - 2p_N)}{\sqrt{2\pi a(1-a)N}} \sim \frac{V_0 C_N}{\sqrt{2\pi a(1-a)N}}, \end{aligned} \quad (23)$$

where the subscript N denotes that the quantities are the values in the case of N particles. We have numerically found that the function $C_N(\tau)$ is approximately the same for any number of particles in this regime of a few particles, and $C_N \sim C$. Thus we have the relation

$$\langle \dot{x}_{\text{cm}} \rangle_N(\tau) \approx \frac{\langle \dot{x} \rangle(\tau)}{\sqrt{2\pi a(1-a)N}} \quad (24)$$

between the velocities for one and for N particles for a given delay τ . This Eq. (24) gives good results for small values of the delay. In particular, inserting Eq. (14) in Eq. (24) we obtain an analytical approximate expression for the case of few particles in the regime of small delays.

We stress that, analogously to the zero delay case [8], the main effect of having a collective ratchet is a decrease in the magnitude of the force fluctuations. This fact gives a center-of-mass velocity inversely proportional to the square-root of the number of particles, as Eq. (24) states. Thereby, if the number N of particles increases,

there will be a decrease of the values of the delay that give better performances for the delayed instant maximization protocol than for the optimal periodic protocol.

On the other hand, for large time delays the analogy between the delayed protocol and the noisy channel protocol no longer gives a good estimate. In this regime of large delays the covariance $\tilde{C}(\tau) = C(\tau) - \mu^2$ becomes negligible indicating that $\text{sgn}f(t - \tau)$ and $\text{sgn}f(t)$ are nearly uncorrelated and that the system effectively behaves as if it were driven by an effective open-loop control protocol. In addition, we observe that the value of the center-of-mass velocity becomes independent of the number of particles (see Fig. 7). This is a hallmark of collective open-loop control ratchets, in which the absence of feedback decouples the Langevin equations provided the particles do not explicitly interact with each other.

V. MANY PARTICLES

We study here the effects of time delays in the feedback controlled Brownian ratchet described in Sec. II for the many particle case, considering both the ‘smooth’ potential and the ‘sawtooth’ potential for various potential heights and different initial conditions.

We find that the system presents two regimes separated by a delay τ_{\min} for which the center-of-mass velocity has a minimum; see Fig. 8. In the small delay regime ($\tau < \tau_{\min}$) the flux decreases with increasing delays as one could expect. On the contrary, in the large delay regime ($\tau > \tau_{\min}$) we have observed and explained a surprising effect, namely, the center-of-mass velocity increases for increasing delays and the system presents several stable solutions. We have found that this critical time delay τ_{\min} is inversely proportional to the potential height $\tau_{\min} \propto 1/V_0$ with a proportionality constant that mildly depends on the number of particles.

A. Zero delay

The many particle ratchet in absence of delay (i.e., $\tau = 0$ in the model of Sec. II) has been studied in Ref. [8]. It has been shown that the net force per particle exhibits a quasideterministic behavior that alternates large periods of time t_{on} with $f(t) > 0$ (on dynamics) and large periods of time t_{off} with $f(t) < 0$ (off dynamics). The center-of-mass velocity can be computed as

$$\langle \dot{x}_{\text{cm}} \rangle = \frac{\Delta x(t_{\text{on}})}{t_{\text{on}} + t_{\text{off}}}, \quad (25)$$

with

$$\Delta x(t_{\text{on}}) = \Delta x_{\text{on}}[1 - e^{-t_{\text{on}}/(2\Delta t_{\text{on}})}], \quad (26)$$

where Δx_{on} and Δt_{on} are obtained fitting the displacement during the ‘on’ evolution for an infinite number of particles (see Ref. [10] for details).

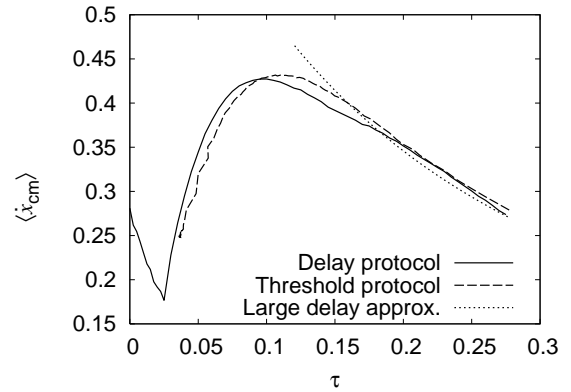


FIG. 8: Many particle case: Center-of-mass velocity as a function of the delay (for large delays only the first branch is represented here), and comparison with the results obtained with the threshold protocol and with the large delay approximation Eq. (33). For the ‘smooth’ potential (5) with $V_0 = 5k_B T$ and $N = 10^5$ particles. Units: $L = 1$, $D = 1$, $k_B T = 1$.

On the other hand, for many particles the fluctuations of the net force are smaller than the maximum value of the net force (see Fig. 9). This allows the decomposition of the dynamics as the dynamics for an infinite number of particles plus the effects of the fluctuations due to the finite value of N . The late time behavior of the net force $f(t)$ for an infinite number of particles is given for the on and off dynamics by [8]

$$f_{\nu}^{\infty}(t) = C_{\nu} e^{-\lambda_{\nu}(t-\tau_{\nu})} \text{ with } \nu = \text{on, off}. \quad (27)$$

The coefficients C_{ν} , λ_{ν} , and τ_{ν} can be obtained fitting this expression with the results obtained integrating a mean field Fokker-Planck equation obtained in the limit $N \rightarrow \infty$ and without delay; see Refs. [8, 10] for details. For a finite number of particles the fluctuations in the force induce switches of the potential and the times on and off are computed equating f_{ν}^{∞} to the amplitude of the force fluctuations, resulting in [8]

$$t_{\text{on}} + t_{\text{off}} = b + d \ln N, \quad (28)$$

with $b = C_{\text{on}} + C_{\text{off}}$ and $d = (\lambda_{\text{on}} + \lambda_{\text{off}})/(2\lambda_{\text{on}}\lambda_{\text{off}})$.

B. Small delays

For small delays, $\tau < \tau_{\min}$, we observe that the flux decreases with the delay. See Fig. 8. We have seen that this decrease is slower than that found for the few particle case (Sec. IV), and that the expressions derived to describe this decrease in the few particle case does not hold here. However, the decrease observed here can be understood noting that a change in the sign of $f(t)$ is perceived by the controller a time τ after, what delays the reaction

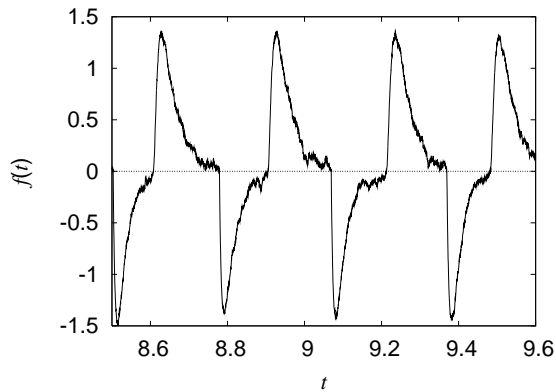


FIG. 9: Many particle case: Evolution of the net force with zero delay ($\tau = 0$) for the ‘smooth’ potential Eq. (5) with $V_0 = 5k_B T$ and $N = 10^5$ particles. Units: $L = 1$, $D = 1$, $k_B T = 1$.

of the system and makes the tails of $f(t)$ longer and implies an increase of the time interval between switches. In addition, the form of $f(t)$ is less smooth than for no delay because the delayed reaction of the controller allows to have several sign flips in the $f(t)$ tails before the system reacts. This sign flips give short epochs of fast switches of the potential (between long on and off epochs), which lead to large fluctuations in $f(t)$. These large fluctuations eventually destabilize these long period solutions for $\tau \sim \tau_{\min}$. See Fig. 10.

As the main effect of the delay is to stretch the ‘on’ and ‘off’ times of the dynamics, using the many particle approximation [8] we can write

$$\langle \dot{x}_{\text{cm}} \rangle = \frac{\Delta x_{\text{on}}}{t_{\text{on}} + t_{\text{off}} + \Delta\tau} = \frac{\Delta x_{\text{on}}}{b + d \ln N + \Delta\tau}, \quad (29)$$

where we have found that the increase of the length of the on-off cycle $\Delta\tau$ is proportional to the delay $\Delta\tau \propto \tau$.

C. Large delays

After the minimum flux is reached for $\tau = \tau_{\min}$, the flux begins to increase with the time delay (see Fig. 8). This increase is due to a change in the dynamical regime: for $\tau > \tau_{\min}$ the present net force starts to be nearly synchronized with the net force a time τ ago. This *self-synchronization* gives rise to a quasiperiodic solution of period $\mathcal{T} = \tau$. Note that there is not a strict periodicity due to stochastic fluctuations in the ‘on’ and ‘off’ times. Looking at the $f(t)$ dependence, Fig. 11, we see that the solutions stabilized by the self-synchronization are similar to those obtained with the threshold protocol [9, 10]. In Fig. 8 we show that the threshold protocol that has the same period gives similar center-of-mass velocity values, confirming the picture. (Differences are due to the fact that we have considered for the threshold protocol

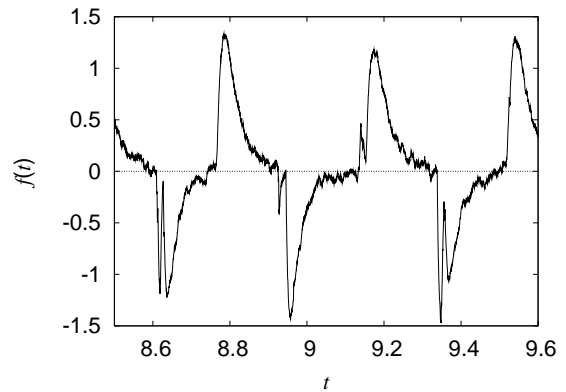


FIG. 10: Many particle case: Evolution of the net force with a small delay ($\tau = 0.02$) for the ‘smooth’ potential Eq. (5) with $V_0 = 5k_B T$ and $N = 10^5$ particles. Units: $L = 1$, $D = 1$, $k_B T = 1$.

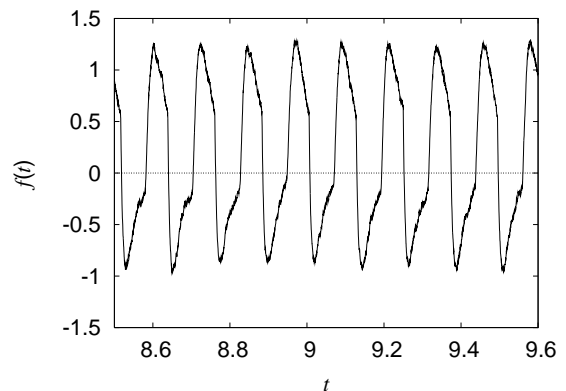


FIG. 11: Many particle case: Evolution of the net force with a large delay ($\tau = 0.12$) for the ‘smooth’ potential Eq. (5) with $V_0 = 5k_B T$ and $N = 10^5$ particles. Units: $L = 1$, $D = 1$, $k_B T = 1$.

simulations with on and off thresholds of the same magnitude, while Fig. 11 shows that the effective thresholds are different.)

This picture allows one to understand the increase of velocity for increasing delay, and the presence of a maximum. This maximum is related with the optimal values of the thresholds that have been shown in [10] to give a quasiperiodic solution of period $\mathcal{T} = \mathcal{T}_{\text{on}} + \mathcal{T}_{\text{off}}$, with \mathcal{T}_{on} and \mathcal{T}_{off} the optimal ‘on’ and ‘off’ times of the periodic protocol. Therefore if we know the values of \mathcal{T}_{on} and \mathcal{T}_{off} for the optimal periodic protocol [$\mathcal{T}_{\text{on}} \sim (1 - a)^2/V_0$ and $\mathcal{T}_{\text{off}} \sim a^2/2$] we can predict that the maximum of the center-of-mass velocity is reached for a delay

$$\tau_{\text{max}} = \mathcal{T}_{\text{on}} + \mathcal{T}_{\text{off}}, \quad (30)$$

and has a value

$$\langle \dot{x}_{\text{cm}} \rangle_{\text{closed}}(\tau_{\text{max}}) = \langle \dot{x}_{\text{cm}} \rangle_{\text{open}}^{\text{max}}, \quad (31)$$

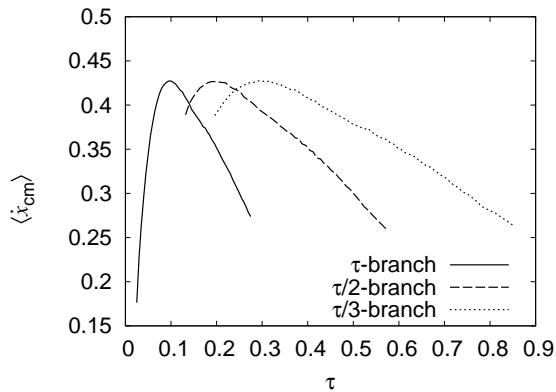


FIG. 12: Many particle case: First three branches of stable solutions for the ‘smooth’ potential (5) with $V_0 = 5k_B T$ and $N = 10^5$ particles. Units: $L = 1$, $D = 1$, $k_B T = 1$.

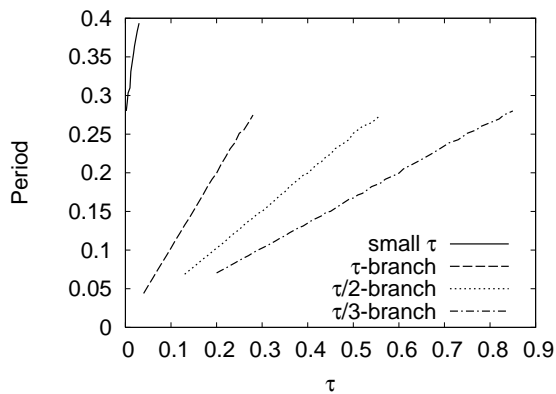


FIG. 13: Many particle case: Period \mathcal{T} of the quasiperiodic solutions for small delays and for large delays. For large delays only the first three branches of stable solutions are shown. ‘Smooth’ potential (5) with $V_0 = 5k_B T$ and $N = 10^5$ particles. Units: $L = 1$, $D = 1$, $k_B T = 1$.

with $\langle \dot{x}_{\text{cm}} \rangle_{\text{open}}^{\text{max}}$ the center-of-mass velocity for the optimal open-loop protocol. Thus this expression gives the position and height of the maximum of the delayed feedback control protocol in terms of the characteristic values of the optimal open-loop control. In particular, it implies that the position and height of the maximum for the flux is independent of the number of particles.

As an example we can apply these expressions to the ‘smooth’ potential with $V_0 = 5$ that for the optimal periodic protocol gives $\langle \dot{x}_{\text{cm}} \rangle = 0.44$ for $\mathcal{T}_{\text{on}} = 0.06$ and $\mathcal{T}_{\text{off}} = 0.05$, so we obtain $\tau_{\text{max}} = 0.06 + 0.05 = 0.11$ in agreement with Fig. 8.

For values of the delay of the order of or larger than τ_{max} quasiperiodic solutions of other periods start to be

stable; see Fig. 12. The periods for the net force $f(t)$ that are found are those that fit an integer number of periods inside a time interval τ , verifying that the present net force is synchronized with the net force a time τ ago, that is, the quasiperiodic solutions have periods $\mathcal{T} = \tau/2$, $\mathcal{T} = \tau/3, \dots$. In addition, it can be seen that the center-of-mass velocity of the n branch $\langle \dot{x}_{\text{cm}} \rangle_{\tau/n}$ whose $f(t)$ has period $\mathcal{T} = \tau/n$ is related with that of the $\mathcal{T} = \tau$ branch through

$$\langle \dot{x}_{\text{cm}} \rangle_{\tau/n}(\tau) = \langle \dot{x}_{\text{cm}} \rangle_{\tau}(\tau/n). \quad (32)$$

We highlight that several branches can be stable for the same time delay τ . Whether the system finally goes to one or another stable solution depends on the initial conditions and on the particular realization of the noise. See Figs. 12 and 13. For these branches we have found initial conditions that go to these solutions and that remain in them during several thousands of periods, indicating that they are stable solutions or at least metastable solutions with a large lifetime.

The analogy with the threshold protocol allows one to use the analytic results of [10] to get further insight in the numerical results. The behavior for large delays for the $\mathcal{T} = \tau$ branch can be obtained using the relation

$$\langle \dot{x}_{\text{cm}} \rangle = \frac{\Delta x(\tau)}{\tau}, \quad (33)$$

with $\Delta x(\tau)$ given by Eq. (26). This equation gives a good prediction for the largest delays of the first branch (see Fig. 8).

On the other hand, for very large values of the delays of the first branch the solutions in a given branch start to become unstable, which can be understood noting that this happens when the fluctuations of the net force become of the order of the absolute value of the net force. Thus the maximum delay that gives a stable solution in the first branch is

$$\tau_{\text{inst}} = t_{\text{on}} + t_{\text{off}} = b + d \ln N, \quad (34)$$

where b and d are determined as in Eq. (28). For example, for the ‘smooth’ potential with $V_0 = 5$, which has $b = -0.070$ and $d = 0.031$, we obtain for $N = 10^5$ particles the value $\tau_{\text{inst}} = 0.29$ in accordance with the numerical results shown in Figs. 8 and 12.

The previous results for the first branch, Eqs. (33) and (34), can be extended to other branches by direct application of the relation (32).

VI. CONCLUSIONS

In this paper we have faced a fundamental question intrinsically related with feedback Brownian ratchets, namely, the effects of a time delay in such a feedback controlled stochastic system. We have focused on the task of studying the dependence of the flux with the time delay

for both the case of one particle and for the collective version of the ratchet with few particles.

For *one particle* ratchets and small delays we have obtained an effective potential which contains the basic ingredients that come into play, and gives an approximate analytical expression for the flux. The effects of the delay in the shape and the average slant of the effective potential allows one to easily understand the decrease of the flux with increasing delays. The approximate analytical expression obtained [Eqs. (14) and (15)] gives the average velocity in terms of the main magnitudes of the system: the height of the potential V_0 , its asymmetry a , and the time delay in the feedback τ . In particular, it allows one to obtain predictions of the characteristic time scale of the decrease due to the delay. This relation is also useful in the *few particle* case thanks to the relation (24) found between the flux for the one and the few particle cases.

The decrease of the covariance of the sign of the net force for increasing delays provides an alternative approach to understand the dependence of the flux with the delay. This approach has given the relation between the covariance and the flux, and has allowed us to relate the flux obtained in the few particle case with the results of the one particle case [Eq. (24)]. In addition, the fact that the covariance becomes negligible for large delays indicates that the delayed control protocol effectively behaves as if it were an open-loop control protocol. This results in a constant value of the flux for large delays that is independent on the number of particles.

We want to stress as an important result of this paper that the feedback controlled system for one or few particles is able to perform better than its open-loop counterpart even for nonzero time delays (provided the delays are smaller than the characteristic times of the dynamics of the Brownian ratchet). Furthermore, even for arbitrarily large delays the net flux does not vanish but it reaches a positive value, albeit it performs worse than the optimal open-loop protocol. We also highlight the importance of this study for realistic *experimental* situations that necessarily have to face with time delays. For the ratchet considered in [11] the colloidal particles have diameter 0.25, 0.4, and 1 μm , and the sawtooth dielectric potential has period $L = 50 \mu\text{m}$ and asymmetry $a \sim 1/3$. The maximum velocities obtained with a periodic switching were reported [11] to be of 0.2 $\mu\text{m/s}$ with $\mathcal{T}_{\text{on}} \sim 30 \text{ s}$ and $\mathcal{T}_{\text{off}} \sim 50 \text{ s}$. As the trapping energy is significantly greater than kT and $a \sim 1/3$ the introduction of feedback can increase the velocity up to a factor $(1/2 - a)^{-1} \sim 6$ approximately [8, 13], attained when the time delay in the feedback is negligible. The results obtained in this paper indicate that for delays in the feedback smaller than the characteristic times of the system (of order 10 s, *i.e.*, of order 10^{-3} in the adimensional units used throughout our paper) it is possible to obtain velocities greater than the maximum of open-loop protocols. The use of a conventional CCD camera (30 fps) and conventional electronics is enough to achieve a feedback control performance with a time delay of the order of 0.1 s (10^{-4} in adimensional

units), for this time delay an increase of the velocity of a factor of 4 is expected. This points towards the feasibility of implementing experimentally a feedback controlled ratchet that performs better than its optimal open-loop version.

We have also studied the effects of time delays in the *many particle* case, where surprising and interesting results arise. Although in the many particle case without delay the instantaneous maximization protocol performs worse than the optimal open-loop protocol, the introduction of a delay can increase the center-of-mass velocity up to the values given by the optimal open-loop control protocol. For small delays the asymptotic average velocity decreases for increasing delays, until it reaches a minimum. After this minimum, a change of regime happens and the system enters a self-synchronized dynamics with the net force at present highly correlated with the delayed value of the net force used by the controller. This self-synchronization stabilizes several of the quasiperiodic solutions that can fit an integer number of periods in a time interval of the length of the time delay. The stable quasiperiodic solutions have a structure similar to those solutions appearing in the threshold protocol. This analogy has allowed us to make numerical and analytical predictions using the previous results for the threshold protocol [10]. In particular, we have established the location and value of the maximum, and also the value of the time delay beyond which a quasiperiodic solution becomes unstable. The results obtained show that for most time delays several solutions are stable and therefore the systems present multistability; which stable solution is reached depends on the past history of the system. The possibility to choose the quasiperiod of the solution we want to stabilize just tuning the time delay can have potential applications to easily control the particle flux. Note that we can even leave some branch just going to time delays where the branch is already unstable, and force the system to change to another branch of solutions.

In summary, we have studied the effects of time delays in the feedback control of a flashing ratchet. The results for one and few particles point towards the feasibility of an experimental implementation of a feedback controlled ratchet that performs better than its optimal open-loop version. On the other hand, the many particle case presents an unexpected improvement of the flux due to the stabilization of one or more quasiperiodic solutions for large enough delays.

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